# A Defect Correction Scheme for the Accurate Evaluation of Magnetic Fields on Unstructured Grids

Ulrich Römer[1,2], Sebastian Schöps[1,2], Herbert De Gersem[1]

[1]Institut für Theorie Elektromagnetischer Felder, Technische Universität Darmstadt, D-64289 Darmstadt, Germany, roemer@temf.tu-darmstadt.de, degersem@temf.tu-darmstadt.de

[2]Graduate School of Computational Engineering, Technische Universität Darmstadt, D-64293 Darmstadt, Germany, schoeps@gsc.ce.tu-darmstadt.de

**In many low-frequency field simulations one is interested in a highly accurate evaluation of the field distribution in an observer region. We propose defect correction as an easy to implement and efficient alternative to higher order finite elements or hybrid approaches. Commonly splines have been used on structured grids for the reconstruction of the solution. Here, we introduce the use of radial basis functions on unstructured grids and study the convergence on the basis of an academic example.**

***Index Terms*—Finite element analysis, Fourier series, interpolation, magnetostatics**

## I. Introduction

In many low-frequency field simulations, e.g., of bending or focusing magnets used in particle accelerators, one is only interested in the magnetic field distribution within an *observer region*. Here, one typically aims at locally uniform magnetic fields, quantified by multipole coefficients (Fourier harmonics) [1]. These quantities of interest need to be evaluated with high accuracy, as undesired multipoles of even small magnitude might influence the device's performance. As the solution is smooth in those regions a local $p$-adaptive finite element approach is a very efficient strategy to this end. Moreover, several dedicated schemes have been presented in the literature: we mention a BEM-FEM coupling [2], a hybrid finite element, spectral element scheme described in [3], and [4] where an improved field gradient was obtained by a local post-processing based on an analytical solution.

However, the approaches mentioned require significant code modifications. Here, we investigate another approach, referred to as defect correction. It is also based on a reconstructed solution. However, additionally a part of the numerical error is estimated and removed from the solution to obtain an improved result. Defect correction is often applied on structured grids based on a tensor-product spline reconstruction, see [5]. The case of an unstructured grid did not receive much attention so far. In [5] biharmonic smoothing was presented and analyzed. As solvers for the biharmonic equation are typically unavailable in a computational magnetics context we present a radial basis function reconstruction as an alternative in this paper. Numerical results are given to illustrate the accuracy of the defect corrected multipoles.

## II. Problem Formulation

To focus on the defect correction principles we consider a two-dimensional, magnetostatic setup. Let the computational domain $\Omega$ be composed of a ferromagnetic and non-ferromagnetic domain $\Omega_{\mathrm{fer}}$ and $\Omega_0$, respectively. The magnetic reluctivity is assumed to be constant $\nu(\vec{x},\cdot) = \nu_0$, for $\vec{x} \in \Omega_0$ and a function of the magnetic field $\nu(\vec{x},|\vec{B}|) = \nu(|\vec{B}|)$, for $\vec{x} \in \Omega_{\mathrm{fer}}$. For simplicity we omit the explicit dependency on $\vec{x}$ from now on. Applying the Newton-Raphson method to the magnetostatic problem, we obtain for the $k$-th iterate $u^{(k)}$

$$- \nabla \cdot \left( \boldsymbol{\nu}_{\mathrm{L}}(\vec{B}^{(k-1)}) \nabla u^{(k)} \right) = f(\vec{B}^{(k-1)}), \qquad (1)$$

in $\Omega$, together with suitable interface and Dirichlet boundary conditions. In (1) $u$ refers to the remaining component of the magnetic vector potential and $f$ contains both source terms and contributions of the magnetic flux density at step $k-1$; $\boldsymbol{\nu}_{\mathrm{L}}$ refers to the tensor valued linearized reluctivity. The associated weak formulation reads, find $u \in V = H_0^1(\Omega)$ such that

$$\int_\Omega \boldsymbol{\nu}_{\mathrm{L}}(\vec{B}^{(k-1)}) \nabla u^{(k)} \cdot \nabla v \, \mathrm{d}x = \int_\Omega f(\vec{B}^{(k-1)}) v \, \mathrm{d}x, \qquad (2)$$

for all $v \in V$. Multipole coefficients are extracted from $u$ or $\vec{B}$, in a small region $\Omega_{\mathrm{obs}} \subset \Omega_0$. Following [1, p.243] in a local polar coordinate system at a reference radius $r_{\mathrm{ref}}$

$$u(r_{\mathrm{ref}}, \varphi) = \sum_{n=1}^{\infty} \left( \mathcal{F}_n \cos(n\varphi) + \mathcal{E}_n \sin(n\varphi) \right). \qquad (3)$$

The coefficients $\mathcal{F}_n$ and $\mathcal{E}_n$ are referred to as normal and skew multipole coefficients, respectively. In the following we focus on the normal multipole coefficients solely. As $\mathcal{F}_n$ at $r_{\mathrm{ref}}$ is given by

$$\mathcal{F}_n(u) = \frac{1}{\pi} \int_0^{2\pi} u(r_{\mathrm{ref}}, \varphi) \cos(n\varphi) \, \mathrm{d}x, \qquad (4)$$

each coefficient is a linear functional of the solution. In practice they can be obtained using a discrete Fourier transform on a circle of radius $r_{\mathrm{ref}}$. Finally, we discretize (2) by finite elements, i.e., look for $u_h \in V_h$ the space of lowest order nodal elements on a triangular mesh. Then the multipole coefficients $\mathcal{F}_{h,n}$ are obtained by replacing $u$ with $u_h$ in (4).

## III. Defect Correction

The fundamental idea is to interpolate the low-order numerical solution between the finite element nodes $(\vec{x}_i)_{i=1}^N$ to obtain a higher order reconstruction of the true solution. Following [5], given a reconstructed solution $\pi^* u_h^{(k)} = u_h^{(k)*}$, we solve for the correction $e_h^{(k)} \in V_h$ subject to

$$\int_\Omega \boldsymbol{\nu}_{\mathrm{L}}(\vec{B}_h^{(k-1)}) \nabla e_h^{(k)} \cdot \nabla v_h \, \mathrm{d}x = \int_\Omega f(\vec{B}_h^{(k-1)}) v_h \, \mathrm{d}x - \int_\Omega \boldsymbol{\nu}_{\mathrm{L}}(\vec{B}_h^{(k-1)}) \nabla u_h^{(k)*} \cdot \nabla v_h \, \mathrm{d}x, \quad \forall v_h \in V_h, \quad (5)$$

to obtain an improved solution $\pi^*(u_h^{(k)} + e_h^{(k)})$ and hence improved multipole coefficients. Defect correction can thus be regarded as an inexact Newton step [6]. The integrals appearing on the right-hand side of equation (5) are evaluated using a Gaussian quadrature of degree two.

On a structured grid, bivariate cubic spline interpolation can be used. On an unstructured grid we determine a radial basis function, more precisely a thin plate spline

$$\pi^* u_h(\vec{x}) = \sum_{i=1}^N \alpha_i |\vec{x} - \vec{x}_i|^2 \log(|\vec{x} - \vec{x}_i|) + p(\vec{x}), \qquad (6)$$

such that $\pi^* u_h(\vec{x}_i) = u_h(\vec{x}_i), \quad \forall i = 1, \ldots, N$, where $p$ is a polynomial function to achieve uniqueness. Due to the non-local support of radial basis functions, the determination of the $\alpha_i$ requires the solution of a dense system of equations. However, as we are interested in the error in $\Omega_{\mathrm{obs}}$ solely, the reconstruction can be restricted to $\Omega_{\mathrm{obs}}$. If in addition, the approximation of the geometry and its singularities is accurate enough, the overall discretization error will be efficiently reduced. Convergence rates for the defect correction scheme have been given in [5]. In particular for smooth solutions, defect correction based on cubic splines yields an $\mathcal{O}(h^4)$ error decay for the multipole coefficients. In the case of thin plate splines we analyze the convergence rate numerically in the following section.

## IV. Numerical Example

For illustration we consider an academic example. The results are obtained using the open-source software FEniCS [7]. On the domain $\Omega = [-1, 1]^2$ with constant linear reluctivity and vanishing current density, we obtain an ideal (normal) sextupole field $u = x^3 - 3xy^2$ by taking $u$ as a non-homogeneous Dirichlet boundary condition. At a reference radius $r_{\mathrm{ref}} = 0.2$ we numerically approximate $\mathcal{F}_3 = 0.008$. Starting with an unstructured grid of mesh size $h := 1/\sqrt{N} = 0.125$, which is aligned at the reference circle, several steps of uniform mesh refinement are carried out. The multipole coefficients are computed performing an FFT along the reference circle. The results are depicted in Figure 1. We observe an improved convergence rate from $h^{1.96}$ to $h^{3.05}$ for the thin plate spline defect corrected multipoles.

## V. Conclusion and Outlook

We have described how to improve the accuracy in the numerically evaluated multipole coefficients by defect correction. Carrying out an additional (inexact) Newton step after post-processing the finite element solution, an improved convergence rate could be observed for an example on unstructured grids. On structured grids more accurate results can be expected by a bivariate spline reconstruction [5].

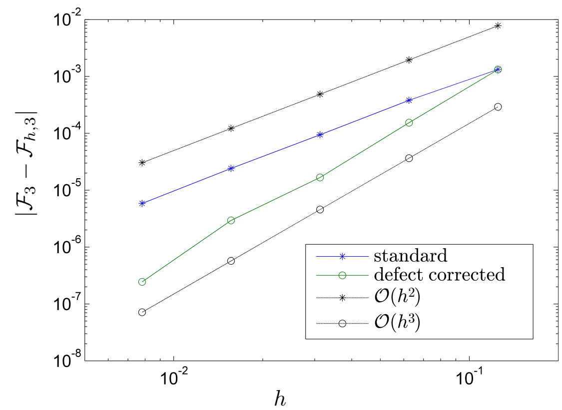


Fig. 1. Discretization error in sextupole component $\mathcal{F}_3$ computed with standard lowest order finite elements and defect correction using thin plate splines. The error decay with respect to the mesh size is improved from approximately $h^2$ to $h^3$.

The scheme requires only minor modifications of the computational code. As the size of the dense system of equations to be solved can be kept small, we expect to obtain an improved overall numerical complexity with respect to standard lowest order finite elements.

A complexity analysis and a precise description of the local reconstruction is subject of ongoing work. Also a theoretical investigation for the convergence rates obtained by the thin plate spline reconstruction is carried out. Finally, the use of adjoint correction techniques [5] is a promising tool to further enhance the defect correction capabilities.


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